Applied Mathematics and Computation 363 (2019) 124549

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Anti-k-labeling of graphs

Xiaxia Guan[a], Shurong Zhang[a], Rong-hua Li[b], Lin Chen[c], Weihua Yang[a,*]

[a] Department of Mathematics, Taiyuan University of Technology, Shanxi, Taiyuan 030024, China
[b] School of Computer Science & Technology, Beijing Institute of Technology, Beijing 100081, China
[c] Lab. de Recherche en Informatique (LRI), CNRS, University of Paris-Sud XI, Orsay 91405, France

ARTICLE INFO

Keywords:
Anti-k-labeling problem
No-hole anti-k-labeling number
Trees
Channel assignment problem,

ABSTRACT

It is well known that the labeling problems of graphs arise in many (but not limited to) networking and telecommunication contexts. In this paper we introduce the anti-k-labeling problem of graphs which we seek to minimize the similarity (or distance) of neighboring nodes. For example, in the fundamental frequency assignment problem in wireless networks where each node is assigned a frequency, it is usually desirable to limit or minimize the frequency gap between neighboring nodes so as to limit interference.

Let $k \geq 1$ be an integer and ψ is a labeling function (anti-k-labeling) from $V(G)$ to $\{1, 2, \ldots, k\}$ for a graph G. A *no-hole anti-k-labeling* is an anti-k-labeling using all labels between 1 and k. We define $w_\psi(e) = |\psi(u) - \psi(v)|$ for an edge $e = uv$ and $w_\psi(G) = \min\{w_\psi(e) : e \in E(G)\}$ for an anti-k-labeling ψ of the graph G. *The anti-k-labeling number* of a graph G, $\lambda_k(G)$, is $\max\{w_\psi(G) : \psi\}$. In this paper, we first show that $\lambda_k(G) = \lfloor \frac{k-1}{\chi-1} \rfloor$, and the problem that determines the anti-k-labeling number of graphs is NP-hard. We mainly obtain the lower bounds on no-hole anti-n-labeling number for trees, grids and n-cubes.

1. Problems

All graphs considered here are simple and finite. Definitions which are not given here may be found in [1]. Let $k \geq 1$ be an integer. An *anti-k-labeling* ψ of a graph G is a mapping from $V(G)$ to $\{1, 2, \ldots, k\}$. An anti-k-labeling ψ of G is called a *no-hole anti-k-labeling* if it uses all labels between 1 and k. We define $w_\psi(e) = |\psi(u) - \psi(v)|$ ($w^{nh}_\psi(e) = |\psi(u) - \psi(v)|$) for an edge $e = uv$ and $w_\psi(G) = \min\{w_\psi(e) : e \in E(G)\}$ ($w^{nh}_\psi(G) = \min\{w^{nh}_\psi(e) : e \in E(G)\}$) for an anti-$k$-labeling ψ (a no-hole anti-k-labeling ψ) of the graph G. The *anti-k-labeling number* (*the no-hole anti-k-labeling number*) of a graph G, $\lambda_k(G)$ ($\lambda^{nh}_k(G)$), is $\max\{w_\psi(G) : \psi\}$ ($\max\{w^{nh}_\psi(G) : \psi\}$). We refer to a labeling ψ with $w_\psi(G) = \lambda_k(G)$ ($w^{nh}_\psi(G) = \lambda^{nh}_k(G)$) as an *optimal anti-$k$-labeling* (an *optimal no-hole anti-k-labeling*) for a graph G. Such (no-hole) anti-k-labeling number problem is our focus in this paper.

The above labeling problem represents a generic class of labeling problems arising in many (but not limited to) networking and telecommunication contexts, in which we seek to minimize the similarity (or distance) of neighboring nodes. For example, in the fundamental frequency assignment problem in wireless networks where each node is assigned a frequency, it is usually desirable to limit or minimize the frequency gap between neighboring nodes so as to limit interference. Another example relates to the content sharing systems such as peer-to-peer file sharing systems, where resources (e.g., files) are

* Corresponding author.
E-mail address: yangweihua@tyut.edu.cn (W. Yang).

https://doi.org/10.1016/j.amc.2019.06.063
0096-3003/

replicated at network nodes to reduce resource retrieval time and increase system robustness. In these systems, to maximize performance gain, we usually want to place different items in the vicinity of each node or to place the same items far from each other.

These problems can be cast to the labeling problem where we seek a node labeling maximizing the minimum labeling distance among neighboring nodes. Surprisingly, this labeling problem has not yet been analyzed (not even formulated in a mathematical sense).

Let T be a set of nonnegative integers. Find a function f: $V(G) \to Z^+$ such that $|f(x)-f(y)| \notin T$ for $xy \in E(G)$. This function f is called a *T-coloring* of G. The span under f is $max\{|f(x)-f(y)| : x, y \in V(G)\}$. We denote the minimum span over all T-colorings by $sp_T(G)$. If $T = \{0, 1, \cdots, m-1\}$, then this T-coloring is called an *m-distant coloring*. Moreover, if all colors are used, then this m-distant coloring is called a *no-hole m-distant coloring*. When $m = 1$, then an m-distant coloring is an ordinary graph coloring. Hence, m-distant coloring is a generalization of ordinary graph coloring.

In some sense, our focus problem is also m-distant coloring. In fact, $\lambda_k(G) > 0$ if and only if $k \geq \chi(G)$ for a graph G, where $\chi(G)$ is the chromatic number of the graph G. Hence, $\chi(G)$ is the minimum number of k such that $\lambda_k(G) > 0$ for a graph G. Since determining the chromatic number of graphs is NP-hard, the anti-k-labeling problem is also NP-hard.

Another related labeling problem (namely, $L(2, 1)$-labeling) will be mentioned in Section 4.

2. $\lambda_k(G)$ and $\chi(G)$ of graphs

Observation 1. If H is a subgraph of G, then $\lambda_k(H) \geq \lambda_k(G)$.

Proof. Clearly, for an arbitrary anti-k-labeling ψ, $w_\psi(H) \geq w_\psi(G)$ holds. Suppose ψ is an optimal anti-k-labeling of G (i.e., $w_\psi(G) = \lambda_k(G)$), then $w_\psi(H) \geq w_\psi(G) = \lambda_k(G)$. Hence, $\lambda_k(H) \geq \lambda_k(G)$ by the definition of anti-k-labeling number. □

Suppose that G_1 and G_2 are two graphs with $V(G_1) \cap V(G_2) = \emptyset$. The *union* G of G_1 and G_2, denoted by $G = G_1 \cup G_2$, is the graph whose vertex set is $V(G_1) \cup V(G_2)$, and edge set is $E(G_1) \cup E(G_2)$.

Observation 2. If $G = G_1 \cup G_2$, then $\lambda_k(G) = \min\{\lambda_k(G_1), \lambda_k(G_2)\}$.

Proof. $\lambda_k(G) \leq \min\{\lambda_k(G_1), \lambda_k(G_2)\}$ following from Observation 1 and the fact that G_1 and G_2 are subgraphs of $G_1 \cup G_2$. On the other hand, an anti-k-labeling of G_1 together with an anti-k-labeling of G_2 makes an anti-k-labeling ψ of $G_1 \cup G_2$ so that $\omega_\psi(G) \geq \min\{\lambda_k(G_1), \lambda_k(G_2)\}$. Hence $\lambda_k(G) \geq \min\{\lambda_k(G_1), \lambda_k(G_2)\}$. □

Theorem 3. *Let G be a graph with chromatic number $\chi = \chi(G) \geq 2$. Then $\lambda_k(G) = \lfloor \frac{k-1}{\chi-1} \rfloor$ for all k.*

Proof. We first show that $\lambda_k(G) \geq \lfloor \frac{k-1}{\chi-1} \rfloor$. It suffices to show that there exists an anti-k-labeling ψ such that $w_\psi(G) = \lfloor \frac{k-1}{\chi-1} \rfloor$ for a graph G. Let $V_1, V_2, \ldots, V_\chi$ be a proper χ-coloring of G. Then we consider the following labeling ψ: label the vertices of V_i by $1 + (i-1)\lfloor \frac{k-1}{\chi-1} \rfloor$, $i = 1, 2, \ldots, \chi$. Note that $1 + (\chi-1)\lfloor \frac{k-1}{\chi-1} \rfloor \leq k$ and V_i $(i = 1, 2, \ldots, \chi)$ is an independent set. We have $w_\psi(G) = \min\{w_\psi(e) : e \in E(G)\} = \lfloor \frac{k-1}{\chi-1} \rfloor$. Hence, $\lambda_k(G) \geq \lfloor \frac{k-1}{\chi-1} \rfloor$.

We next show that $\lambda_k(G) \leq \lfloor \frac{k-1}{\chi-1} \rfloor$. Let ψ be an optimal anti-k-labeling of G and $(V_1, V_2, \ldots, V_k)$ be a partition of $V(G)$ under ψ, where the vertices in V_i have label i, $i = 1, 2, \ldots, k$. Assume $\lambda_k(G) \geq \lfloor \frac{k-1}{\chi-1} \rfloor + 1$. We colour the vertices of $V_{(i-1)\lfloor \frac{k-1}{\chi-1} \rfloor + i}, V_{(i-1)\lfloor \frac{k-1}{\chi-1} \rfloor + i+1}, \ldots, V_{i\lfloor \frac{k-1}{\chi-1} \rfloor + i}$ with color c_i $(i = 1, 2, \ldots, \chi-2)$, and color the vertices of $V_{(\chi-2)\lfloor \frac{k-1}{\chi-1} \rfloor + \chi-1}, V_{(\chi-2)\lfloor \frac{k-1}{(\chi-1)} \rfloor + \chi}, \ldots, V_k$ with color $c_{\chi-1}$. Note that $k \leq (\chi-1)(\lfloor \frac{k-1}{\chi-1} \rfloor) + \chi - 1$. And the vertices of V_i are not adjacent to the vertices of V_j $(1 \leq j \leq k)$, $j \in \{i - \lfloor \frac{k-1}{\chi-1} \rfloor, i - \lfloor \frac{k-1}{\chi-1} \rfloor + 1, \ldots, i + \lfloor \frac{k-1}{\chi-1} \rfloor\}$ by the assumption $\lambda_k(G) \geq \lfloor \frac{k-1}{\chi-1} \rfloor + 1$. Thus, the vertices of coloring c_i $(i = 1, 2, \ldots, \chi-1)$ are not adjacent. This implies a proper $(\chi-1)$-coloring of G, a contradiction. Therefore $\lambda_k(G) = \lfloor \frac{k-1}{\chi-1} \rfloor$. □

By Theorem 3, $\lambda_{k'}(G) \geq \lambda_k(G)$ holds for $k' \geq k$. And for some integer k, if $\lambda_k(G) = m$, then $\frac{k-1}{m+1} + 1 < \chi(G) \leq \frac{k-1}{m} + 1$. In particular, if k is the minimum number with $\lambda_k(G) = m$, then $\chi = \frac{k-1}{m} + 1$. This is line with the following Theorem.

Theorem 4 [5]**.** *$sp_T(G) = m(\chi-1)$ for $T = \{0, 1, \cdots, m-1\}$.*

3. $\lambda_n^{nh}(G)$ of graphs

In this section we consider no-hole anti-k-labeling for $k = n$.

Observation 5. If G' is a spanning subgraph of G, then $\lambda_n^{nh}(G') \geq \lambda_n^{nh}(G)$.

Proof. Suppose $\lambda_n^{nh}(G) = l$ with an optimal labeling ψ. Let $w_\psi^{nh}(G') = w_\psi^{nh}(e)$. Then $\lambda_n^{nh}(G') \geq w_\psi^{nh}(G') = w_\psi^{nh}(e) \geq w_\psi^{nh}(G) = l$ by the definitions. Therefore, $\lambda_n^{nh}(G') \geq l$. □

Observation 6. For a graph G with n vertices, $\lambda_n(G) \geq \lambda_n^{nh}(G)$ holds for all $n \geq 2$.

Proof. It is obvious that $\lambda_n(G) \geq \lambda_n^{nh}(G)$. $\square$

We denote by δ and Δ the minimum degree and maximum degree of a graph G. We have the following.

Observation 7. For a connected graph G with n vertices, $\lambda_n^{nh}(G) \geq 1$ and $\lambda_n^{nh}(G) \leq \min\{n-\Delta, \lfloor \frac{n-1}{\chi-1} \rfloor, \lfloor \frac{n-\delta+1}{2} \rfloor\}$ hold for all $n \geq 2$.

Proof. For each no-hole anti-n-labeling ψ, $w_\psi^{nh}(G) \geq 1$. Thus, $\lambda_n^{nh}(G) \geq 1$.

Note that the vertex with the maximum degree has Δ neighbors which have distinct labels for any no-hole anti-n-labeling. Then $\lambda_n^{nh}(G) \leq n-\Delta$.

Let v be the vertex having label $\lceil \frac{n}{2} \rceil$ for an optimal no-hole anti-n-labeling ψ of G, then there is an edge e incident to v so that $w_\psi^{nh}(e) \leq \lfloor \frac{n-\delta+1}{2} \rfloor$ since there are at least δ vertices adjacent to v in G. Therefore $\lambda_n^{nh}(G) \leq \lfloor \frac{n-\delta+1}{2} \rfloor$.

It is clear that $\lambda_n^{nh}(G) \leq \lambda_n(G) = \lfloor \frac{n-1}{\chi-1} \rfloor$ by Observation 6 and Theorem 3. Thus, the claim holds. $\square$

Theorem 8 [8]**.** *For a graph G, $\lambda_n^{nh}(G) \geq n$ if and only if G has no edges.*

Let G be a simple graph. The complement graph G^c of G is the simple graph with vertex set $V(G)$, two vertices being adjacent in G^c if and only if they are not adjacent in G. An m-path with $m' > m$ vertices is a sequence of m' distinct vertices of G, $v_1, v_2, \cdots, v_{m'}$, where $v_i, v_{i+1}, \cdots, v_{i+m}$ form a clique ($i = 1, 2, \cdots, m'-m$). An m-path with $m' \leq m$ vertices is simply a clique of order m'. A Hamilton m-path of G is an m-path containing all vertices of G.

Theorem 9 [8]**.** *For a graph G, $\lambda_n^{nh}(G) \geq m+1$ if and only if there exists a Hamilton m-path for G^c.*

By Theorem 9, one can see that the no-hole anti-n-labeling number implies some structural properties of graphs.

Corollary 10. *For a graph G, $\lambda_n^{nh}(G) \geq 2$ if and only if there exists a Hamilton path for the complement graph G^c of G.*

Proof. This is an immediate consequence of $m = 1$ in Theorem 9. $\square$

Corollary 11. *For a non-empty graph G (i.e., G has at least an edge), $\lambda_n^{nh}(G) \leq \alpha(G)$, where $\alpha(G)$ is the independence number of G.*

Proof. Suppose $\lambda_n^{nh}(G) = m$. Then G^c contains a Hamilton $(m-1)$-path by Theorem 9, and $m < n$ by Theorem 8, since G has at least an edge. And so G^c contains a clique of order m. That is, G has an independent set of order m. Hence, $\alpha(G) \geq m$. $\square$

Next, we consider the no-hole anti-n-labeling number of some special graphs.

3.1. $\lambda_n^{nh}(G)$ of complete multipartite graphs

Theorem 12 [8]**.** *If G contains a complete t-partite subgraph H and $|V(G)| - |V(H)| < (t-1)(m-1)$, then $\lambda_n^{nh}(G) < m$.*

Corollary 13. *Let $K_{n_1,\cdots,n_t}$ be a complete t-partite graph with n vertices. Then $\lambda_n^{nh}(K_{n_1,\cdots,n_t}) = 1$ holds for all $n \geq 2$.*

Proof. It is clear according to Observation 7 and $m = 2$ of Theorem 12. $\square$

We next consider an example for graph operations. Suppose G_1 and G_2 are two graphs with disjoint vertex sets. The *join* G of G_1 and G_2, denoted by $G = G_1 + G_2$, is the graph obtained from $G_1 \cup G_2$ by adding all edges between vertices in $V(G_1)$ and vertices in $V(G_2)$.

Corollary 14. *If $G = G_1 + G_2$, then $\lambda_n^{nh}(G) = 1$.*

Proof. If $G = G_1 + G_2$, then G' is a spanning subgraph of G, where G' is a complete bipartite graph with bipartition ($V(G_1)$, $V(G_2)$). Hence $\lambda_n^{nh}(G) = 1$ by Observation 5 and Corollary 13. $\square$

3.2. $\lambda_n^{nh}(G)$ of trees

Theorem 15. *Let P_n be a path on n vertices. Then $\lambda_n^{nh}(P_n) = \lfloor \frac{n}{2} \rfloor$.*

Proof. Since $\delta(P_n) = 1$, $\lambda_n^{nh}(P_n) \leq \lfloor \frac{n}{2} \rfloor$ according to Observation 7.

Let $v_1, v_2, \ldots, v_n$ be vertices of P_n such that v_i is adjacent to v_{i+1}, $1 \leq i \leq n-1$. Now we show that $\lambda_n^{nh}(P_n) \geq \lfloor \frac{n}{2} \rfloor$. It suffices to show that there is a no-hole anti-n-labeling ψ such that $w_\psi^{nh}(P_n) = \lfloor \frac{n}{2} \rfloor$ for P_n. Consider the following labeling:

(i) If n is even, then we define

$$\psi(v_i) = \begin{cases} \frac{n}{2} - \frac{i-1}{2} & i \text{ is odd}, \\ n+1-\frac{i}{2} & i \text{ is even}. \end{cases}$$

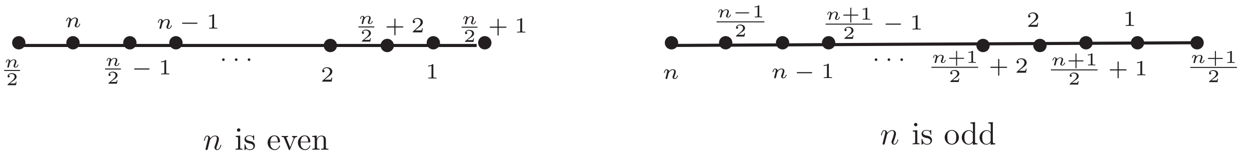

Fig. 1. The labels of paths.

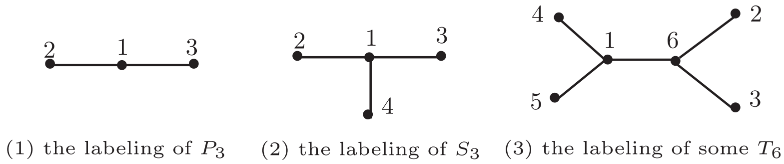

Fig. 2. The labeling of some trees.

(ii) If n is odd, then we define

$$\psi(v_i) = \begin{cases} \frac{n+1}{2} - \frac{i-1}{2} & i \text{ is odd,} \\ n+1-\frac{i}{2} & i \text{ is even.} \end{cases}$$

Clearly, for each $e \in E(P_n)$, $w_\psi(e)$ is $\frac{n}{2}$ or $\frac{n}{2}+1$ for even n, and $\frac{n-1}{2}$ or $\frac{n+1}{2}$ for odd n. Hence $\lambda_n^{nh}(P_n) \geq \lfloor \frac{n}{2} \rfloor$ (Fig. 1). □

We denote by T_n a tree with n vertices. Note that a tree is a bipartite graph. A *leaf* in a tree is a vertex of degree 1.

Lemma 16. *For a tree T_n with bipartition (X_1, X_2), and $|X_1| < |X_2|$, we have X_2 contains a leaf of T_n.*

Proof. By contradiction, suppose that X_2 contains no leaves of T_n. Let $Y_0 = \{u : d(u) = 1, u \in V(T_n)\}$, $Y_i = \{v : \exists u \in Y_0$ so that $d(u, v) = i\} \setminus \cup_{j=0}^{i-1} Y_j$, $m = \max\{i : Y_i \neq \emptyset\}$. Note $Y_i \subseteq X_1$ ($Y_i \subseteq X_2$, resp.) for even (odd, resp.) $i \leq m$. And X_i is an independent set for $i = 1, 2$. Thus, $|Y_{i+1}| \leq |Y_i|$ $(i = 0, 1, \ldots, m-1)$ due to T_n without cycle.

If m is even, then $|X_2| = |Y_1| + |Y_3| + \ldots + |Y_{m-1}| \leq |Y_0| + |Y_2| + \ldots + |Y_{m-2}| < |Y_0| + |Y_2| + \ldots + |Y_{m-2}| + |Y_m| = |X_1|$, a contradiction. If m is odd, then $|X_2| = |Y_1| + |Y_3| + \ldots + |Y_m| \leq |Y_0| + |Y_2| + \ldots + |Y_{m-1}| = |X_1|$, a contradiction. Thus, X_2 contains a leaf of T_n. □

Theorem 17. *For a tree T_n with bipartition (X_1, X_2), $|X_i| = q_i$, $i = 1, 2$, we have $\lambda_n^{nh}(T_n) \geq q = \min\{q_1, q_2\}$.*

Proof. The result clearly holds for $n = 1, 2$. Without loss of generality, we suppose that $q_1 \leq q_2$ for $n \geq 3$, i.e., $q = q_1$. We show that $\lambda_n^{nh}(T_n) \geq q$ by giving a no-hole anti-n-labeling ψ_n of T_n with $w_{\psi_n}^{nh}(T_n) \geq q$ and $\psi_n(v) \leq q$ ($\psi_n(v) > q$, resp.) for $v \in X_1$ ($v \in X_2$, resp.). If $n = 3$, then $T_3 = P_3$. Let $T_3 = P_3 = v_1v_2v_3$. Then $v_2 \in X_1$ and $v_1, v_3 \in X_2$. Let ψ_3 be the optimal no-hole anti-3-labeling defined in Theorem 15 for T_3. We have $\psi_3(v_1) = 3$, $\psi_3(v_2) = 1$, and $\psi_3(v_3) = 2$, and $w_{\psi_3}^{nh} \geq 1 = q$ according to Theorem 15. Hence, $\lambda_n^{nh}(T_n) \geq q$ for $n = 3$. Moreover, each vertex of X_1 (X_2, resp.) is labeled by $i \leq q$ ($i > q$, resp.) in the labeling ψ_3.

We next construct the no-hole anti-n-labeling ψ_n of T_n by induction on $n \geq 4$. We assume that ψ_m is a no-hole anti-k-labeling of T_k satisfying the requirement for $k < n$. We label T_n based on the labeling ψ_k of T_k as below.

Case 1. $q_1 < q_2$.

By Lemma 16, there exists a leaf $u \in X_2$ of T_n. Let $T_{n-1} = T_n - u$. Clearly, $|X_1(T_{n-1})| = |X_1(T_n)| = q_1 = q$, $|X_2(T_{n-1})| = |X_2(T_n)| - 1 = q_2 - 1$ and $q_1 \leq q_2 - 1$. By the induction hypothesis, there exists a no-hole anti-$(n-1)$-labeling ψ_{n-1} so that $w_{\psi_{n-1}}^{nh}(T_{n-1}) \geq q_1$ and each vertex of $X_1(T_{n-1})$ ($X_2(T_{n-1})$, resp.) is labeled by $i \leq q_1$ ($i > q_1$, resp.). We obtain the labeling ψ_n by labeling the vertex u by n based on ψ_{n-1}. It is obvious that $w_{\psi_n}^{nh}(T_n) \geq q$, and each vertex of $X_1(T_{n-1})$ ($X_2(T_{n-1})$, resp.) is labeled by $i \leq q$ ($i > q$, resp.) in the labeling ψ_n (see Fig. 2(2)).

Case 2. $q_1 = q_2 = q = \frac{n}{2}$.

Clearly, there is a vertex (say u) whose neighbors are all leaves except one vertex for any tree T_n. Without loss of generality, we assume that $u \in X_2$ and u has m leaves as its neighbors. We consider the graph T_{n-m-1} obtained from T_n by removing the vertex u and the m neighbors (the m leaves) of u. Note $|X_1(T_{n-m-1})| = |X_1(T_n)| - m = \frac{n}{2} - m$, $|X_2(T_{n-m-1})| = |X_2(T_n)| - 1 = \frac{n}{2} - 1$. By the induction hypothesis, there exists a no-hole anti-$(n-m-1)$-labeling ψ_{n-m-1} so that $w_{\psi_{n-m-1}}^{nh}(T_{n-m-1}) \geq \frac{n}{2} - m$ and each vertex of $X_1(T_{n-m-1})$ ($X_2(T_{n-m-1})$, resp.) is labeled by $i \leq \frac{n}{2} - m$ ($i > \frac{n}{2} - m$, resp.).

We now label T_n by the following rules (i.e., ψ_n): relabel the vertex with label $i > \frac{n}{2} - m$ in T_{n-m-1} by $i + m$, label the vertex u by n, and label the m neighbors of u by $\frac{n}{2} - m + 1, \frac{n}{2} - m + 2, \cdots, \frac{n}{2}$. Clearly, $\psi_n(v) \leq \frac{n}{2}$ ($\psi_n(v) > \frac{n}{2}$, resp.) for $v \in X_1(T_n)$ ($v \in X_2(T_n)$, resp.) in the labeling ψ_n of T_n.

1	23	6	28	11	33	16	38
21	4	26	9	31	14	36	19
2	24	7	29	12	34	17	39
22	5	27	10	32	15	37	20
3	25	8	30	13	35	18	40

(1) $P_5 \times P_8$

1	14	2	15	3
16	4	17	5	18
6	19	7	20	8
21	9	22	10	23
11	24	12	25	13

(2) $P_5 \times P_5$

Fig. 3. Labels of $P_5 \times P_5$ and $P_5 \times P_8$.

Next we show $w^{nh}_{\psi_n}(T_n) \geq \frac{n}{2}$, i.e., $w^{nh}_{\psi_n}(e) \geq \frac{n}{2}$ for all $e = uv \in E(T_n)$ in ψ_n. If $e \in E(T_{n-m-1})$, then $w^{nh}_{\psi_n}(T_n) \geq \frac{n}{2}$ since $w^{nh}_{\psi_{n-m-1}}(T_{n-m-1}) \geq \frac{n}{2} - m$ by the induction hypothesis and $w^{nh}_{\psi_n}(e) \geq |\psi_n(u) - \psi_n(v)| = |\psi_{n-m-1}(u) - \psi_{n-m-1}(v)| + m$. If $e \notin E(T_{n-m-1})$, then e is incident to u. Note that u is labeled by n and its neighboring vertices are labeled by some integer $i \leq \frac{n}{2}$ in ψ_n. We have $w^{nh}_{\psi_n}(e) \geq \frac{n}{2}$. Hence, $w^{nh}_{\psi_n}(T_n) \geq q$ (see Fig. 2(3)). $\square$

Remark 18. For an arbitrary bipartition (X_1, X_2), $|X_1| = q_1 \leq |X_2| = q_2$, there is a tree T_n such that $\lambda^{nh}_n(T_n) = q_1$. We consider the tree T_n as following: T_n is obtained by joining $q_1 - 1$ new vertices to leaves in the star graph K_{1,q_2}. Since $\Delta(T_n) = q_2$. Then $\lambda^{nh}_n(T_n) \leq n - q_2 = q_1$ by Observation 7. Therefore, $\lambda^{nh}_n(T_n) = q_1$ by Theorem 17.

We also pose a conjecture below.

Conjecture 19. *For a tree T_n with bipartition (X_1, X_2), $X_i = q_i$, $i = 1, 2$, we have $\lambda^{nh}_n(T_n) = q$, where $q = min\{q_1, q_2\}$.*

3.3. $\lambda^{nh}_{mn}(G)$ of 2-Dimensional grids $\mathrm{P_m} \times \mathrm{P_n}$

In this subsection, we generalize the result on paths to 2-Dimensional grids.

Observation 20. Let G is a 2-Dimensional grid $P_m \times P_n (m \leq n)$. Then $\lambda^{nh}_{mn}(G) \geq \lfloor \frac{mn-m}{2} \rfloor$.

Proof. We look the $P_m \times P_n$ grid (i.e., m rows and n columns) as a chessboard. Like in the chessboard, we have white and black alternating squares (see Fig. 3).

(i) If at least one of m and n is even (i.e., mn is even), we have in the “white” squares the labels from the range $[1, \frac{mn}{2}]$ and in the “black” squares the labels from the range $[\frac{mn}{2} + 1, mn]$. Without loss of generality, we assume that the left upper square is white. Take the following labeling ψ: put 1 in the left upper corner (put $\frac{mn}{2} + 1$ in the second square in the first row of grid, resp.) and subsequently put in the white (black, resp.) squares from left to right and row by row the upper range labels: $2, 3, \ldots, \frac{mn}{2}$ ($\frac{mn}{2} + 2, \frac{mn}{2} + 3, \ldots, mn$, resp.).

Let v be labelled by i, $i \leq \frac{mn}{2}$ ($i > \frac{mn}{2}$, resp.). Then the vertices adjacent to v are labelled by $i + \frac{mn}{2}, i + \frac{mn}{2} - 1, i + \lfloor \frac{mn-m}{2} \rfloor, i + \lfloor \frac{mn+m}{2} \rfloor$ ($i - \frac{mn}{2}, i - \frac{mn}{2} + 1, i - \lfloor \frac{mn-m}{2} \rfloor, i - \lfloor \frac{mn+m}{2} \rfloor$, resp.). Hence, $\lambda^{nh}_{mn}(G) \geq \lfloor \frac{mn-m}{2} \rfloor$ (see Fig. 3(1)).

(ii) If m and n are odd (i.e., mn is odd), we have in the “white” squares the labels from the range $[1, \frac{mn+1}{2}]$ and in the “black” squares the labels from the range $[\frac{mn+1}{2} + 1, mn]$. Take the following labeling ψ: put 1 in the left upper corner (put $\frac{mn+1}{2} + 1$ in the second square in the first row of grid, resp.) and subsequently put in the white (black, resp.) squares from left to right and row by row the upper range labels: $2, 3, \ldots, \frac{mn+1}{2}$ ($\frac{mn+1}{2} + 2, \frac{mn+1}{2} + 3, \ldots, mn$, resp). We have $\lambda^{nh}_{mn}(G) \geq \frac{mn-m}{2}$ by the argument of (i) (see Fig. 3(2)). $\square$

Conjecture 21. *Let G is a 2-Dimensional grid $P_m \times P_n$. Then $\lambda^{nh}_{mn}(G) = \lfloor \frac{mn-m}{2} \rfloor$, where $m = \min\{m, n\}$.*

3.4. $\lambda^{nh}_{2^n}(G)$ of n-cubes

Theorem 22. *For a cycle C_n of length n, $\lambda^{nh}_n(C_n) = \lfloor \frac{n-1}{2} \rfloor$ (Fig. 4).*

Proof. Since $\delta(C_n) = 2$, $\lambda^{nh}_n(C_n) \leq \lfloor \frac{n-1}{2} \rfloor$ according to Observation 7.

Now we show that $\lambda^{nh}_n(C_n) \geq \lfloor \frac{n-1}{2} \rfloor$. It suffices to show that there is a labeling ψ such that $w^{nh}_{\psi}(C_n) = \lfloor \frac{n-1}{2} \rfloor$. Let C_n be $v_1 \to v_2 \to \ldots \to v_n \to v_1$. Consider the following labeling:

(i) If n is odd, then we define

$$\psi(v_i) = \begin{cases} \frac{n+1}{2} - \frac{i-1}{2} & i \text{ is odd,} \\ n + 1 - \frac{i}{2} & i \text{ is even.} \end{cases}$$

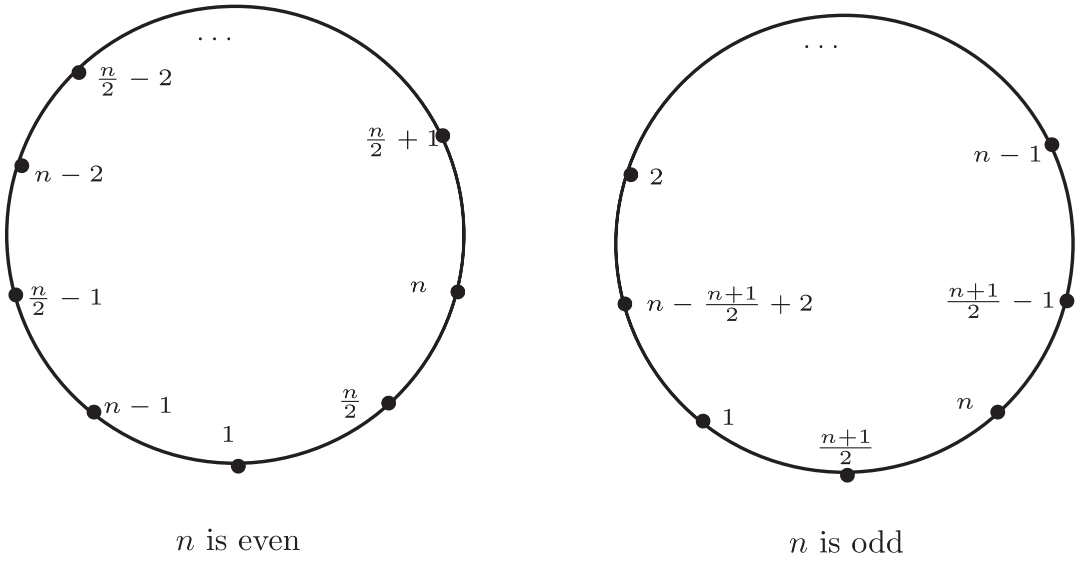

Fig. 4. The labels of cycles.

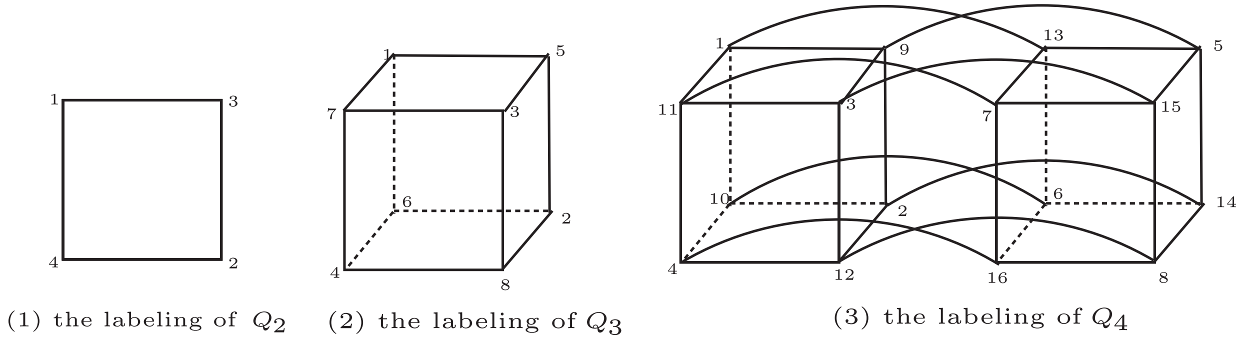

Fig. 5. The labelings of Q_2, Q_3, and Q_4.

(ii) If n is even, then we define

$$\psi(v_i) = \begin{cases} 1 & i \text{ is } 1, \\ n & i \text{ is } 3, \\ \frac{i-1}{2} & i \text{ is odd and } i \neq 1, 3, \\ \frac{n}{2} - 1 + \frac{i}{2} & i \text{ is even.} \end{cases}$$

It is easy to show that $w^{nh}_{\psi}(e)$, $e \in E(C_n)$, defined above is $\frac{n}{2}$ or $\frac{n}{2} - 1$ for even n, and $\frac{n-1}{2}$ or $\frac{n+1}{2}$ for odd n. Hence $\lambda^{nh}_{n}(C_n) \geq \lfloor \frac{n-1}{2} \rfloor$. $\square$

An n-cube can be defined inductively as follows. An 1-cube is a P_2. An n-cube Q_n may be constructed from the disjoint union of two $(n-1)$-cubes Q_{n-1}, by adding an edge from each vertex in one copy of Q_{n-1} to the corresponding vertex in the other copy. The joining edges form a perfect matching.

Theorem 23. *Let Q_n be an n-cube. Then, for all $n \geq 2$, $\lambda^{nh}_{2^n}(Q_n) \geq 2^{n-2}$.*

Proof. We show $\lambda^{nh}_{2^n}(Q_n) \geq 2^{n-2}$ by constructing a no-hole anti-2^n-labeling ψ_n such that $w^{nh}_{\psi_n}(Q_n) \geq 2^{n-2}$, and one end has label at most 2^{n-1} and the other end has label greater 2^{n-1} for each edge in Q_n. If $n = 2$, then $Q_n = C_4$. By Theorem 22, $\lambda^{nh}_{2^2}(Q_2) = 1 \geq 2^{2-2}$. Let ψ_2 be the optimal no-hole anti-2^2-labeling defined in Theorem 22 of Q_2. Clearly, for each edge e of Q_2, one end of e has label at most $2^{2-1} = 2$ and the other end of e has label greater 2 under ψ_2, see Fig. 5(1). For $m \leq n$, we assume there exists a labeling ψ_m such that $w^{nh}_{\psi_m}(Q_m) \geq 2^{m-2}$, and one end has label at most 2^{m-1} and the other end has label greater 2^{m-1} for each edge in Q_m. We next construct the labeling ψ_{n+1} satisfying the assumption above for Q_{n+1} from the labeling ψ_n defined above of Q^1_n and Q^2_n as follows.

Note that an $(n+1)$-cube Q_{n+1} can be obtained by adding a perfect matching PM between two copies of an n-cube, denoted by Q_n^1 and Q_n^2 (Each edge of PM joins two vertices having the same labels.). We relabel the vertices with label $i > 2^{n-1}$ in Q_n^1 by $i+2^{n-1}$, and we relabel the vertices with label $i \leq 2^{n-1}$ in Q_n^2 by $i+2^n+2^{n-1}$.

We next show that the assumption above holds for ψ_{n+1} in Q_{n+1}. Let $e=uv$ be an edge of $E(Q_{n+1})$. We firstly assume $e \in E(Q_n^1)$ and $\psi_n(u) > \psi_n(v)$. By the induction hypothesis, we have $\psi_n(u) > 2^{n-1}$, $\psi_n(v) \leq 2^{n-1}$ and $\psi_n(u) - \psi_n(v) \geq 2^{n-2}$. Therefore $\psi_{n+1}(u) = \psi_n(u) + 2^{n-1} > 2^n$, $\psi_{n+1}(v) = \psi_n(v) \leq 2^{n-1} < 2^n$, and $w^{nh}_{\psi_{n+1}}(e) = |\psi_{n+1}(u) - \psi_{n+1}(v)| = \psi_{n+1}(u) - \psi_{n+1}(v) = \psi_n(u) + 2^{n-1} - \psi_n(v) \geq 2^{n-1} + 2^{n-2} > 2^{n-1}$ according to the definition of ψ_{n+1}. If $e \in E(Q_n^2)$ and we suppose $\psi_n(u) > \psi_n(v)$. Then $\psi_n(u) > 2^{n-1}$, $\psi_n(v) \leq 2^{n-1}$, and $\psi_n(u) - \psi_n(v) < 2^n$. Therefore $\psi_{n+1}(u) = \psi_n(u) < 2^n$, $\psi_{n+1}(v) = \psi_n(v) + 2^n + 2^{n-1} > 2^n$, and $w^{nh}_{\psi_{n+1}}(e) = |\psi_{n+1}(u) - \psi_{n+1}(v)| = \psi_{n+1}(v) - \psi_{n+1}(u) = \psi_n(v) + 2^n + 2^{n-1} - \psi_n(u) > 2^{n-1}$. Finally, we assume $e \in E(PM)$. Without loss of generality, we assume $u \in V(Q_n^1)$ and $v \in V(Q_n^2)$. Then $\psi_n(u) = \psi_n(v)$. If $\psi_n(u) \leq 2^{n-1}$, then $\psi_{n+1}(u) = \psi_n(u) < 2^n$, $\psi_{n+1}(v) = \psi_n(v) + 2^n + 2^{n-1} > 2^n$, and $w^{nh}_{\psi_{n+1}}(e) = 2^n + 2^{n-1}$. If $\psi_n(u) > 2^n$, then $\psi_{n+1}(u) = \psi_n(u) + 2^{n-1} > 2^n$, $\psi_{n+1}(v) = \psi_n(v) < 2^n$, and $w^{nh}_{\psi_{n+1}}(e) = 2^{n-1}$. We complete the proof. □

Theorem 24. *Let Q_3 be a 3-cube. Then $\lambda^{nh}_8(Q_3) = 2$.*

Proof. We have $\lambda^{nh}_8(Q_3) \geq 2$ by Theorem 23. We next show $\lambda^{nh}_8(Q_3) \leq 2$ by contradiction. Suppose $\lambda^{nh}_8(Q_3) \geq 3$. Let ψ be an optimal labeling and we denote by v_i the vertex with label i under ψ. Then v_4 may only be adjacent to vertices v_1, v_7, v_8, v_5 may only be adjacent to vertices v_1, v_2, v_8, and v_6 may only be adjacent to vertices v_1, v_2, v_3 in Q_3 due to $mc^{nh}_8(Q_3) \geq 3$. Note that Q_3 is a bipartite graph. Let the bipartition of Q_3 be (X, Y), and $|X| = |Y| = 2^{3-1} = 4$. Without loss of generality, we assume $v_4 \in X$. Then $v_1, v_7, v_8 \in Y$, and $v_5, v_6 \in X$. Hence, $v_1, v_2, v_3, v_7, v_8 \in Y$, that is, $|Y| = 5$, a contradiction. □

Note that the bound in Theorem 23 is sharp for $n = 2, 3$. $\lambda^{nh}_{2^n}(Q_n) \leq \lfloor \frac{2^n - n + 1}{2} \rfloor$ holds by Observation 7. We pose the following problem.

Conjecture 25. *For all $n \geq 2$, $\lambda^{nh}_{2^n}(Q_n) = 2^{n-2}$.*

4. Anti-L_d(2, 1)-labeling of graphs

Given a simple graph $G = (V, E)$ and a positive number d, an L_d(2, 1)-labeling of G is a function f: $V(G) \to [0, \infty)$ such that whenever $x, y \in V$ are adjacent, if $|f(x) - f(y)| \geq 2d$, and whenever the distance between x and y is two, if $|f(x) - f(y)| \geq d$. The L_d(2, 1)-labeling number $\lambda(G, d)$ is the smallest number m such that G has an L_d(2, 1)-labeling f with $\max\{f(v) : v \in V\} = m$. When $d = 1$, the L_d(2, 1)-labeling problem is the L(2, 1)-labeling problem. The L(2, 1)-labeling problem of graphs has been discussed for many graph families, see [2–4,7,9,10].

Similarly, we define the *anti-L_d(2, 1)-labeling* problem: given a simple graph $G = (V, E)$ and a positive number d, a labeling of G is a function f: $V(G) \to [1, k]$ such that $|f(x) - f(y)| \geq 2d$ if $xy \in E(G)$, $|f(x) - f(y)| \geq d$ if $d(x, y) = 2$. The *anti-L_d(2, 1)-labeling number* of G, denoted by $\lambda^L_k(G)$, is the largest number $2d$.

By the proofs of Observations 1 and 2, we have the results of Observations 26 and 27 as following.

Observation 26. If H is a subgraph of G, then $\lambda^L_k(H) \geq \lambda^L_k(G)$.

Observation 27. If $G = G_1 \cup G_2$, then $\lambda^L_k(G) = \min\{\lambda^L_k(G_1), \lambda^L_k(G_2)\}$.

Lemma 28 [7]**.** *$\lambda(G, d) = d \cdot \lambda(G, 1)$ for a non-negative integer d.*

Lemma 29 [6]**.** *$\lambda(G, 1) \leq \Delta^2 + \Delta - 2$.*

Theorem 30. *Let G is a simple graph. Then $\lambda^L_k(G) \geq 2\lfloor \frac{k-1}{\Delta^2 + \Delta - 2} \rfloor$.*

Proof. Suppose that $\lambda(G, d) = m$ for a graph G. Then there exists a labeling f: $V(G) \to [0, m]$ such that whenever $x, y \in V$ are adjacent, if $|f(x) - f(Y)| \geq 2d$, and whenever the distance between x and y is two, if $|f(x) - f(Y)| \geq d$. Therefore, there exists a labeling $\psi : V(G) \to [1, m+1]$, such that $w_\psi(G) = 2d$ for $k = m + 1 = \lambda(G, d) + 1$. According to Lemma 28, there exists a labeling ψ, such that $w^\lambda_\psi(G) = 2\frac{k-1}{\lambda(G,1)}$ for all k. Therefore $\lambda^L_k(G) \geq 2\lfloor \frac{k-1}{\lambda(G,1)} \rfloor$ for all k according to the definition of the anti-L_d(2, 1)-labeling number $\lambda^L_k(G)$. Combining with Lemma 29, we have $\lambda^L_k(G) \geq 2\lfloor \frac{k-1}{\Delta^2 + \Delta - 2} \rfloor$. □

Theorem 31. *If $\lambda^L_k(G) = 2d$ for a positive number k, then $\frac{k-1}{d+1} < \lambda(G, 1) \leq \frac{k-1}{d}$.*

Proof. Suppose that $\lambda^L_k(G) = 2d$ for a graph G. Then $\lambda(G, d) + 1 \leq k < \lambda(G, d+1) + 1$. In fact, it is obvious that $\lambda(G, d) + 1 \leq k$, since G has an L_d(2, 1)-labeling for all positive number k and $\lambda(G, d)$ is the smallest number m such that G has an L_d(2, 1)-labeling f. Suppose $k \geq \lambda(G, d+1) + 1$. Then there exists a labeling ψ, such that $w^\lambda_\psi(G) = 2(d+1)$. Hence $\lambda^L_k(G) \geq 2(d+1)$ according to the definition of $\lambda^L_k(G)$, a contradiction. Hence, $d \cdot \lambda(G, 1) + 1 \leq k < (d+1) \cdot \lambda(G, 1) + 1$ combining with Lemma 28, that is $\frac{k-1}{d+1} < \lambda(G, 1) \leq \frac{k-1}{d}$. □

Acknowledgments

The research is supported by NSFC (No. 11671296), Research Project Supported by Shanxi Scholarship Council of China, Program for the Innovative Talents of Higher Learning Institutions of Shanxi (PIT).

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Application, North-Holland, new york, 1976.
[2] T. Calamoneri, The $l(2, 1)$-labeling problem on oriented regular grids, Comput. J. 54 (2011) 1869–1875.
[3] G.J. Chang, J.J. Chen, D. Kuo, S.C. Liaw, Distance-two labelings of digraphs, Discret. Appl. Math. 155 (2007) 1007–1013.
[4] G.J. Chang, S.C. Liaw, The $l(2, 1)$-labeling problem on ditrees, Ars Comb. 66 (2003) 23–31.
[5] M.B. Cozzens, F.S. Roberts, t-colorings of graphs and the channel assignment problem, Congr. Numer. 35 (1982) 191–208.
[6] D. Gonalves, On the $l(p, 1)$-labelling of graphs, Discret. Math. 308 (2008) 1405–1414.
[7] J.R. Griggs, R.K. Yeh, Labelling graphs with a condition at distance 2, SIAM J. Discret. Math. 5 (1992) 586–595.
[8] D. Sakai, C. Wang, No-hole $(r + 1)$-distant colorings, Discret. Math. 119 (1993) 175–189.
[9] R.K. Yeh, Labeling Graphs With a Condition at Distance two, University of South Carolina, Columbia, South Carolina, 1990 Ph.D. Thesis.
[10] R.K. Yeh, A survey on labeling graphs with a condition at distance two, Discret. Math. 306 (2006) 1217–1231.